

Two-connected signed graphs with maximum nullity at most two

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Abstract

A signed graph is a pair (G, Σ) , where $G = (V, E)$ is a graph (in which parallel edges are permitted, but loops are not) with $V = \{1, \dots, n\}$ and $\Sigma \subseteq E$. The edges in Σ are called odd and the other edges of E even. By $S(G, \Sigma)$ we denote the set of all symmetric $n \times n$ matrices $A = [a_{i,j}]$ with $a_{i,j} < 0$ if i and j are adjacent and connected by only even edges, $a_{i,j} > 0$ if i and j are adjacent and connected by only odd edges, $a_{i,j} \in \mathbb{R}$ if i and j are connected by both even and odd edges, $a_{i,j} = 0$ if $i \neq j$ and i and j are non-adjacent, and $a_{i,i} \in \mathbb{R}$ for all vertices i . The parameters $M(G, \Sigma)$ and $\xi(G, \Sigma)$ of a signed graph (G, Σ) are the largest nullity of any matrix $A \in S(G, \Sigma)$ and the largest nullity of any matrix $A \in S(G, \Sigma)$ that has the Strong Arnold Hypothesis, respectively. In a previous paper, we gave a characterization of signed graphs (G, Σ) with $M(G, \Sigma) \leq 1$ and of signed graphs with $\xi(G, \Sigma) \leq 1$. In this paper, we characterize the 2-connected signed graphs (G, Σ) with $M(G, \Sigma) \leq 2$ and the 2-connected signed graphs (G, Σ) with $\xi(G, \Sigma) \leq 2$.

1 Introduction

A *signed graph* is a pair (G, Σ) , where $G = (V, E)$ is a graph (in which parallel edges are permitted, but loops are not) and $\Sigma \subseteq E$. (We refer to [3] for the notions and concepts in graph theory.) The edges in Σ are called *odd* and the other edges *even*. If $V = \{1, 2, \dots, n\}$, we denote by $S(G, \Sigma)$ the set of all real symmetric $n \times n$ matrices $A = [a_{i,j}]$ with

- $a_{i,j} < 0$ if i and j are adjacent and all edges between i and j are even,
- $a_{i,j} > 0$ if i and j are adjacent and all edges between i and j are odd,

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- $a_{i,j} \in \mathbb{R}$ if i and j are connected by odd and even edges,
- $a_{i,j} = 0$ if $i \neq j$ and i and j are non-adjacent, and
- $a_{i,i} \in \mathbb{R}$ for all vertices i .

In [1] we introduced for any signed graph (G, Σ) , among other parameters, the signed graph parameters M and ξ . For a signed graph (G, Σ) , $M(G, \Sigma)$ is the maximum of the nullities of the matrices in $S(G, \Sigma)$. In order to describe the parameter ξ we need the notion of *Strong Arnold Property* (SAP for short). A matrix $A = [a_{i,j}] \in S(G, \Sigma)$ has the SAP if $X = 0$ is the only symmetric matrix $X = [x_{i,j}]$ such that $x_{i,j} = 0$ if i and j are adjacent vertices or $i = j$, and $AX = 0$. Then $\xi(G, \Sigma)$ is defined as the largest nullity of any matrix $A \in S(G, \Sigma)$ satisfying the SAP. It is clear that $\xi(G, \Sigma) \leq M(G, \Sigma)$ for any signed graph (G, Σ) . This signed graph parameter ξ is analogous to the parameter ξ for simple graphs introduced by Barioli, Fallat, and Hogben [2].

If G is a graph and v a vertex of G , then $\delta(v)$ denotes the set of edges of G incident to v . The symmetric difference of two sets A and B is the set $A \Delta B = A \setminus B \cup B \setminus A$. If (G, Σ) is a signed graph and $U \subseteq V(G)$, we say that (G, Σ) and $(G, \Sigma \Delta \delta(U))$ are *sign-equivalent* and call the operation $\Sigma \rightarrow \Sigma \Delta \delta(U)$ *re-signing on U* . Re-signing on U amounts to performing a diagonal similarity on the matrices in $S(G, \Sigma)$, and hence it does not affect $M(G, \Sigma)$ and $\xi(G, \Sigma)$.

Let (G, Σ) be a signed graph. If H is a subgraph of G , then we say that H is *odd* if $\Sigma \cap E(H)$ has an odd number of elements, otherwise we call H *even*. Zaslavsky showed in [7] that two signed graphs are sign-equivalent if and only if they have the same set of odd cycles. Thus, two signed graphs (G, Σ) and (G, Σ') that have the same set of odd cycles have $\xi(G, \Sigma) = \xi(G, \Sigma')$.

In [1], we showed that a signed graph (G, Σ) has $M(G, \Sigma) \leq 1$ if and only if (G, Σ) is sign-equivalent to a signed graph (H, \emptyset) , where H is a graph whose underlying simple graph is a path. Furthermore, we showed that a signed graph (G, Σ) has $\xi(G, \Sigma) \leq 1$ if and only if (G, Σ) is sign-equivalent to the signed graph (H, \emptyset) , where H is a graph whose underlying simple graph is a disjoint union of paths. Observe that in case the signed graph (G, Σ) is connected, $M(G, \Sigma) \leq 1$ if and only if $\xi(G, \Sigma) \leq 1$. In this paper, we characterize the class of 2-connected signed graphs (G, Σ) with $M(G, \Sigma) \leq 2$. We will see that this class coincides with the class of signed graphs (G, Σ) with $\xi(G, \Sigma) \leq 2$.

The above characterizations are extensions of results known for simple graphs to signed graphs. For a simple graph G , denote by $S(G)$ the set of all real symmetric $n \times n$ matrices $A = [a_{i,j}]$ with $a_{i,j} \neq 0$ if i and j are connected by an edge, $a_{i,j} = 0$ if $i \neq j$ and i and j are non-adjacent, and $a_{i,i} \in \mathbb{R}$ for all vertices i . The maximum nullity of a simple graph G is the maximum of the nullities of the matrices in $S(G)$. Fiedler [4] proves that a simple graph G has $M(G) \leq 1$ if and only if G is a path. In [6], Johnson, Loewy, and Smith characterize the class of simple graphs G with $M(G) \leq 2$. Barioli, Fallat, and Hogben introduced in [2] the parameter ξ . For a simple graph G , $\xi(G)$ is defined as the largest nullity of any matrix $A \in S(G)$ satisfying the SAP. In [2], they prove that a graph G

has $\xi(G) \leq 1$ if and only if G is a subgraph of a path. In [5], Hogben and van der Holst give a characterization of the class of simple graphs G with $\xi(G) \leq 2$.

2 The maximum nullity of some signed graphs

Contracting an edge e with ends u and v in a graph G means deleting e and identifying the vertices u and v . A signed graph (H, Γ) is a *weak minor* of a signed graph (G, Σ) if (H, Γ) can be obtained from (G, Σ) by deleting edges and vertices, contracting edges, and re-signing around vertices. We use weak minor to distinguish it from minor in which only even edges are allowed to be contracted. The parameter ξ has the nice property that if (H, Γ) is a weak minor of the signed graph (G, Σ) , then $\xi(H, \Gamma) \leq \xi(G, \Sigma)$.

Let us now introduce some signed graphs. By K_n^e and K_n^o we denote the signed graphs (K_n, \emptyset) and $(K_n, E(K_n))$, respectively. By K_4^i we denote the signed graph $(K_4, \{e\})$, where e is an edge of K_4 . By $K_{2,3}^e$ and $K_{2,3}^i$, we denote the signed graphs $(K_{2,3}, \emptyset)$ and $(K_{2,3}, \{e\})$, where e is an edge of $K_{2,3}$, respectively.

From Proposition 8 in [1], the following lemma follows.

Lemma 1. $M(K_n^e) = \xi(K_n^e) = n - 1$ and $M(K_n^o) = \xi(K_n^o) = n - 1$.

The following lemma follows from Proposition 34 in [1].

Lemma 2. $M(K_4^i) = \xi(K_4^i) = 2$.

The following lemma follows from Proposition 35 in [1].

Lemma 3. $M(K_{2,3}^e) = \xi(K_{2,3}^e) = 3$ and $M(K_{2,3}^i) = \xi(K_{2,3}^i) = 2$.

From Lemmas 1 and 3, it follows that signed graphs (G, Σ) with $\xi(G, \Sigma) \leq 2$ cannot have a weak K_4^e -, K_4^o -, or $K_{2,3}^e$ -minor. If a signed graph (G, Σ) has no weak K_4^e -, K_4^o -, or $K_{2,3}^e$ -minor, the graph G can still have a K_4 - or $K_{2,3}$ -minor. However, these minors force the signed graph to have additional structure. We will study this in the next section.

By W_4 we denote the graph obtained from C_4 by adding a new vertex v and connecting it to each vertex of C_4 . The subgraph C_4 in W_4 is called the *rim* of W_4 . Any edge between v and a vertex of the rim of W_4 is called a *spoke*. Let e_1, e_2 be two nonadjacent edges of the C_4 in W_4 . By W_4^o , we denote the signed graph $(W_4, \{e_1, e_2\})$. See Figure 1 for a picture of W_4^o ; here a bold edge is an odd edge and a thin edge an even edge. This signed graph appears as a special case in the characterization of 2-connected signed graphs (G, Σ) with $M(G, \Sigma) \leq 2$.

Lemma 4. $M(W_4^o) = \xi(W_4^o) = 2$

Proof. Let v_1, v_2, v_3, v_4 be the vertices of the rim of W_4 in this cyclic order, and let v be the hub of W_4 . We assume that the edges between v_1 and v_2 and between v_3 and v_4 are even. Suppose for a contradiction that $M(W_4^o) \geq 3$. Then there exists a matrix $A = [a_{i,j}] \in S(W_4^o)$ with nullity ≥ 3 . Since $\text{nullity}(A) \geq 3$, there

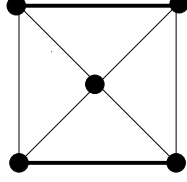


Figure 1: The signed four-wheel.

exist nonzero vectors $x, y \in \ker(A)$ with $x_{v_1} = x_{v_2} = 0$ and $y_{v_1} = y_{v_4} = 0$. If $x_v = 0$, then from $Ax = 0$ it would follow that $x_{v_4} = x_{v_3} = 0$. This contradiction shows that $x_v \neq 0$. We may assume that $x_v > 0$. Then, since the edge between v and v_1 is even, the edge between v_1 and v_4 is odd, and $Ax = 0$, it follows that $x_{v_4} > 0$. Similarly, $x_{v_3} > 0$. Let a_v denote the row of A corresponding to v . Then $a_v x = 0$, and, since $x_v > 0$, $x_{v_4} > 0$, $x_{v_3} > 0$, it follows that $a_{v,v} > 0$.

We will now do the same with the vector y . If $y_v = 0$, then it would follow that $y_{v_2} = y_{v_3} = 0$. This contradiction shows that $y_v \neq 0$. We may assume that $y_v > 0$. Then, since the edge between v_1 and v_2 is even, the edge between v_1 and v is even, and $Ay = 0$, it follows that $y_{v_2} < 0$. Similarly, $y_{v_3} < 0$. Since $a_v y = 0$ and $y_v > 0$, $y_{v_2} < 0$, $y_{v_3} < 0$, it follows that $a_{v,v} < 0$. We have arrived at a contradiction, and we can conclude that $\xi(W_4^o) \leq M(W_4^o) \leq 2$. Since W_4^o contains an odd cycle, $M(W_4^o) \geq \xi(W_4^o) \geq \xi(K_2^-) = 2$, and hence $M(W_4^o) = \xi(W_4^o) = 2$. \square

3 Wide separations

Let (G, Σ) be a signed graph. A pair $[G_1, G_2]$ of subgraphs of G is a *wide separation* of (G, Σ) if there exists an odd 4-cycle C_4 such that $G_1 \cup C_4 \cup G_2 = G$, $E(G_1) \cap E(C_4) = \emptyset$, $E(G_2) \cap E(C_4) = \emptyset$, $V(G_1) \cap V(G_2) = \emptyset$, $V(G_1) \cap V(C_4) = \{r_1, r_2\}$ and $V(G_2) \cap V(C_4) = \{s_1, s_2\}$, where r_1 and r_2 are nonadjacent vertices of C_4 and s_1 and s_2 are nonadjacent vertices of C_4 . We call r_1, r_2 the vertices of attachment of G_1 and s_1, s_2 the vertices of attachment of G_2 in the wide separation.

Lemma 5. *Let (G, Σ) be a signed graph with no weak minor isomorphic to K_4^e , K_4^o , or $K_{2,3}^i$. If G has a K_4 -minor, but no W_4 -minor, then (G, Σ) has a wide separation.*

Proof. Since G has a K_4 -minor and all vertices of K_4 have degree three, G has a subgraph isomorphic to a subdivision of K_4 . Hence there are distinct vertices v_1, v_2, v_3, v_4 and openly disjoint paths P_1, \dots, P_6 of length ≥ 1 in G , where P_1 has ends v_1 and v_2 , P_2 has ends v_1 and v_3 , P_3 has ends v_1 and v_4 , P_4 has ends v_2 and v_3 , P_5 has ends v_2 and v_4 , and P_6 has ends v_3 and v_4 . Let H be the subgraph spanned by v_1, v_2, v_3, v_4 and P_1, \dots, P_6 . Since (G, Σ) has no weak K_4^e - or K_4^o -minor, $(H, \Sigma \cap E(H))$ has no weak K_4^e - or K_4^o -minor.

By re-signing if needed, we may assume that, in (G, Σ) , P_1 is an odd path and that P_2, \dots, P_6 are even paths.

The paths P_1 , P_2 , P_5 , and P_6 consist of single edges.

P_6 is a single edge, for otherwise, possibly after re-signing, we contract an odd edge in P_6 and obtain a signed graph that contains K_4^o as a weak minor. P_1 is a single edge, for otherwise, we contract an odd edge in P_1 and obtain a signed graph that contains K_4^e as a weak minor. If both P_3 and P_5 have length ≥ 2 , then, possibly after re-signing, we contract an odd edge in P_3 and in P_5 , and obtain a signed graph that has K_4^o as a weak minor. Hence at least one of P_3 and P_5 consists of a single edge. In the same way, at least one of P_2 and P_4 consists of a single edge. If both P_2 and P_3 have length ≥ 2 , then possibly after re-signing, we contract an odd edge in P_2 and in P_3 , and obtain a signed graph that has K_4^o as a weak minor. Hence at least one of P_2 and P_3 consists of a single edge. In the same way, at least one of P_4 and P_5 consists of a single edge. If one of P_2 and P_5 has length ≥ 2 , then P_3 and P_4 consist each of a single edge. Hence, we can conclude that both P_2 and P_5 consist of single edges, or that both P_3 and P_4 consist of single edges. By symmetric we may assume that P_2 and P_5 consist of single edges.

Since G has no W_4 -minor, each path connecting a vertex of P_3 to a vertex of P_4 must contain at least one vertex of $\{v_1, v_4\}$ and at least one vertex of $\{v_2, v_3\}$. Thus (G, Σ) has a wide separation. \square

Lemma 6. *Let (G, Σ) be a graph with no weak $K_{2,3}^e$ -minor. If G has $K_{2,3}$ -minor, then (G, Σ) has a wide separation $[G_1, G_2]$, where G_1 is isomorphic to K_2^e .*

Proof. Since G has a $K_{2,3}$ -minor, G has a subgraph H isomorphic to a subdivision of $K_{2,3}$. Hence there are vertices v_1, v_2, v_3, v_4, v_5 and openly disjoint paths P_1, \dots, P_6 of length ≥ 1 in H , where P_1 has ends v_1 and v_2 , P_2 has ends v_1 and v_4 , P_3 has ends v_1 and v_5 , P_4 has ends v_3 and v_4 , P_5 has ends v_2 and v_3 , and P_6 has ends v_3 and v_5 . We now view the paths P_1, \dots, P_6 as paths in the signed graph $(H, \Sigma \cap E(H))$. We may re-sign $(H, \Sigma \cap E(H))$ such that P_1 is odd and P_2, \dots, P_6 are odd.

P_1 is a single edge, for otherwise, possibly after re-signing, we contract an odd edge in P_1 and obtain a signed graph that contains $K_{2,3}^e$ as a weak minor. Similarly, P_5 is a single edge. If both P_2 and P_3 have length ≥ 2 , then, possibly after re-signing, we contract an odd edge in P_2 and in P_3 , and obtain a signed graph that has $K_{2,3}^e$ as a weak minor. Hence at least one of P_2 and P_3 consists of a single edge. Similarly at least one of P_4 and P_6 consists of a single edge, at least one of P_2 and P_6 consists of a single edge, and at least one of P_3 and P_4 consists of a single edge. Hence at most one path of P_2, P_3, P_4, P_6 has length ≥ 2 . That is, (G, Σ) has a wide separation $[G_1, G_2]$, where G_1 is isomorphic to K_2^e . \square

4 The signed four-wheel

In this section we show that if (G, Σ) is a signed graph with no K_4^e -, K_4^o -, or $K_{2,3}^e$ -minor, but G has a W_4 -minor, then the edges in each parallel class of (G, Σ) have the same parity and, after removing from each parallel class all but one edge, we obtain W_4^o .

Lemma 7. *Let (W_4, Σ) be a signed graph with no K_4^e -, K_4^o -, or $K_{2,3}^e$ -minor. Then (W_4, Σ) is sign-equivalent to W_4^o .*

Proof. If at most one triangle of (W_4, Σ) is even, then (W_4, Σ) has a K_4^o -minor. If at most one triangle of (W_4, Σ) is odd, then (W_4, Σ) has a K_4^e -minor. So we may assume that (W_4, Σ) has exactly two odd triangles. If they share an edge, then (W_4, Σ) has a $K_{2,3}^e$ -minor. If they do not share an edge, then (W_4, Σ) is sign-equivalent to W_4^o . \square

Lemma 8. *Let (G, Σ) be obtained from W_4^o by adding an odd or even edge between nonadjacent vertices. Then (G, Σ) has a weak K_4^o - or K_4^e -minor.*

Proof. Let v_1, v_2, v_3, v_4 be the vertices on the rim of W_4 in this order. Up to symmetry there is only one possibility to add an edge between two nonadjacent vertices in W_4 . We may assume that we add the edge e between v_1 and v_3 . If e is an even edge, then the resulting signed graph has a weak K_4^o -minor. If e is an odd edge, then the resulting graph has a weak K_4^e -minor. \square

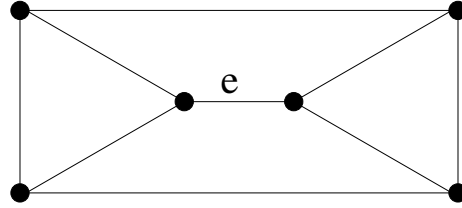


Figure 2: The prism with the edge e

Lemma 9. *Let (G, Σ) be a signed graph which has an edge whose contraction yields W_4^o , then (G, Σ) has a weak K_4^o - or K_4^e -minor.*

Proof. If G has a vertex of degree two, then G arises from W_4 by inserting a new vertex on an edge e of W_4 , which results in a path P of length two in G . If e is an even edge on the rim of W_4 , then, possibly after re-signing, we contract all but one odd edge of P . The resulting signed graph contains a weak K_4^o -minor. If e is an odd edge on the rim of W_4 , then, possibly after re-signing, we contract all but one even edge of P . The resulting signed graph contains a weak K_4^e -minor. If e is a spoke of W_4 , then, possibly after re-signing, we contract all but one odd edge of P . The resulting signed graph contains a weak K_4^o -minor.

So we may assume that G has no vertex of degree two. Then G is isomorphic either to the prism, C_6^e (see Figure 2), or to $K_{3,3}$. Let e be the edge in G such that contracting e yields W_4 .

If G is isomorphic to $K_{3,3}$, then $(G, \Sigma) - e$ contains a weak K_4^o -minor. Suppose next that G is isomorphic to the prism. If (G, Σ) has two odd triangles, then, possibly after re-signing, we contract e and another edge not on the two odd triangles, and obtain a signed graph that contains a K_4^o as a signed subgraph. If (G, Σ) has two even triangles, then, possibly after re-signing, we contract e and another edge not on the two even triangles, and obtain a signed graph that contains a K_4^e as a signed subgraph. \square

Lemma 10. *Let (G, Σ) be a signed graph. If G has a W_4 -minor, then at least one of the following holds:*

1. (G, Σ) has a weak K_4^o -, K_4^e -, or $K_{2,3}^e$ -minor, or
2. the edges in each parallel class of (G, Σ) have the same parity and, after removing from each parallel class all but one edge, we obtain W_4^o .

Proof. Suppose for a contradiction that there exists a signed graph (G, Σ) such that G has a W_4 -minor, but (G, Σ) has no weak K_4^o -, K_4^e -, or $K_{2,3}^e$ -minor and it is not the case that the edges in each parallel class of (G, Σ) have the same parity and, after removing from each parallel class all but one edge, we obtain W_4^o . We take (G, Σ) with $|V(G)| + |E(G)|$ as small as possible.

If the underlying simple graph of G is isomorphic to W_4 , then there must be parallel edges of different parity in (G, Σ) . In this case (G, Σ) has a weak K_4^o -, K_4^e -, or $K_{2,3}^e$ -minor by Lemma 7. So we may assume that the underlying simple graph of G is not isomorphic to W_4 . Since G has a W_4 -minor and G is connected, there exists a signed graph (H, Ω) such that (H, Ω) is a minor of (G, Σ) , the underlying simple graph of H is not isomorphic to W_4 , and W_4^o can be obtained from (H, Ω) by deleting or contracting one edge. If W_4^o can be obtained from (H, Ω) by deleting an edge e , then e connects nonadjacent vertices of W_4^o . In this case, (H, Ω) has a weak K_4^o - or K_4^e -minor by Lemma 8, and hence (G, Σ) has a K_4^o - or K_4^e -minor. If W_4^o can be obtained from (H, Ω) by contracting an edge e , then (H, Ω) has a weak K_4^o - or K_4^e -minor by Lemma 9, and hence (G, Σ) has a weak K_4^o - or K_4^e -minor. \square

5 Partial wide 2-path

A *sided 2-path* is defined recursively as follows:

1. Let T be a triangle and let \mathcal{F} be a set of two distinct edges in this triangle. Then (T, \mathcal{F}) is a sided 2-path.
2. If (G, \mathcal{F}) is a sided 2-path and H is obtained from G by adding edges parallel to the edges in \mathcal{F} , then (H, \mathcal{F}) is a sided 2-path.

3. Let (G, \mathcal{F}) be a sided proper 2-path and let e and f be distinct edges in a disjoint triangle T . If H is obtained from G by identifying the edge f of T with an edge h in \mathcal{F} , then $(H, (\mathcal{F} \setminus \{h\}) \cup \{e\})$ is a sided 2-path.

The edges in \mathcal{F} are called the sides of the sided 2-path. A *2-path* is a graph G for which there exists a set \mathcal{F} of two distinct edges of G such that (G, \mathcal{F}) is a sided 2-path. A *partial 2-path* is a subgraph of a 2-path. A 2-connected partial 2-path with no parallel edges is the same as a linear singly edge articulated cycle graph (LSEAC), a type of graphs introduced by Johnson et al. [6], and it is the same as a linear 2-tree, a type of graphs introduced by Hogben and van der Holst [5].

Lemma 11. *Let G be a 2-connected graph with no K_4 -, $K_{2,3}$ -, or K_3^- -minor. Then G is a partial 2-path.*

Proof. Since G has no K_4 - and no $K_{2,3}$ -minor, G is outerplanar. Hence G can be embedded in the plane such that all its vertices are incident to the infinite face. Add edges such that all finite faces are either triangles or 2-cycles and let the resulting graph be H . Construct the following tree R . The vertices of R are all finite faces of the plane embedding. Connect two vertices of the tree if the corresponding faces have an edge in common. Then R is a path. For if not, there would be a face that has edges in common with at least three other faces. Such a graph has a K_3^- -minor. By induction it can now be shown that H is a 2-path. Hence G is a partial 2-path. \square

A pair $\{e, f\}$ of nonadjacent edges in K_4^i is called *split* if both e and f belong to an even and an odd triangle.

A *sided wide 2-path* is defined recursively as follows:

1. Let (G, Σ) be an even or odd triangle or a K_4^i . If (G, Σ) is a triangle, let \mathcal{F} be distinct edges in this triangle. If $(G, \Sigma) = K_4^i$, let \mathcal{F} be a split pair of edges in K_4^i .
2. If $[(G, \Sigma), \mathcal{F}]$ is a sided wide 2-path and (H, Ω) is obtained from (G, Σ) by adding odd and even edges parallel to edges in \mathcal{F} , then $[(H, \Omega), \mathcal{F}]$ is a sided wide 2-path.
3. Let $[(G, \Sigma), \mathcal{F}]$ be a sided wide 2-path and let e and f be distinct edges in an even or odd triangle T . If (H, Ω) is obtained from (G, Σ) by identifying the edge f of T with an edge h in \mathcal{F} , then $[(H, \Omega), (\mathcal{F} \setminus \{h\}) \cup \{e\}]$ is a sided wide 2-path.
4. Let $[(G, \Sigma), \mathcal{F}]$ be a sided wide 2-path and let $\{e, f\}$ be a split pair of edges in K_4^i . If (H, Ω) is obtained from (G, Σ) by identifying the edge f of K_4^i with an edge h in \mathcal{F} , then $[(H, \Omega), (\mathcal{F} \setminus \{h\}) \cup \{e\}]$ is a sided wide 2-path.

The edges in \mathcal{F} are called the sides of the sided wide 2-path. A *wide 2-path* is a signed graph (G, Σ) for which there exists a set \mathcal{F} of two distinct edges of (G, Σ) such that $[(G, \Sigma), \mathcal{F}]$ is a sided wide 2-path. A signed graph (G, Σ) is a *partial wide 2-path* if it is a spanning subgraph of a wide 2-path.

Lemma 12. *Let (G, Σ) be a 2-connected signed graph. If (G, Σ) has no minor isomorphic to K_4^e , K_4^o , $K_{2,3}^e$, or K_3^- , then, after removing in each parallel class all but one edge of the same parity, (G, Σ) is either isomorphic to W_4^o or (G, Σ) is a partial wide 2-path.*

Proof. If G has a W_4 -minor, then (G, Σ) is isomorphic to W_4^o , by Lemma 10. We may therefore assume that G has no W_4 -minor.

Suppose G has a K_4 -minor. Then, by Lemma 5, (G, Σ) has a wide separation $[G_1, G_2]$. For $i = 1, 2$, let (H_i, Ω_i) be obtained from $(G_i, E(G_i) \cap \Sigma)$ by adding between the vertices of attachments u and v of $(G_i, E(G_i) \cap \Sigma)$ in the wide separation $[G_1, G_2]$ an odd and even edge in parallel. Then (H_i, Ω_i) , $i = 1, 2$, contain no weak minor isomorphic to K_4^e , K_4^o , $K_{2,3}^e$, K_3^- , or W_4^o , for otherwise (G, Σ) would contain a weak minor isomorphic to K_4^e , K_4^o , $K_{2,3}^e$, K_3^- , or W_4^o .

Suppose $G_1 - \{u, v\}$ contains more than one component; let C_1, \dots, C_k be the components. Then, as G is 2-connected, each $G_1[V(C_i) \cup \{u, v\}]$ contains a path of length ≥ 2 connecting u and v . If there are components C_j and C_k such that both $G_1[V(C_j) \cup \{u, v\}]$ and $G_1[V(C_k) \cup \{u, v\}]$ contain an even path connecting u and v , then (G, Σ) has a $K_{2,3}^e$ -minor, a contradiction. Similarly, there are no two components C_j and C_k such that both $G_1[V(C_j) \cup \{u, v\}]$ and $G_1[V(C_k) \cup \{u, v\}]$ contain an odd path connecting u and v . Hence $G_1 - \{u, v\}$ has exactly two components C_1 and C_2 . If $G[V(C_1) \cup \{u, v\}]$ contains a path P of length ≥ 3 connecting u and v and $G[V(C_2) \cup \{u, v\}]$ contains an even path connecting u and v , then, possibly after re-signing, we contract an edge of P to obtain an even path. Then (G, Σ) has a $K_{2,3}^e$ -minor. The cases where $G[V(C_2) \cup \{u, v\}]$ has an odd path connecting u and v , and where $G[V(C_2) \cup \{u, v\}]$ has a path of length ≥ 3 connecting u and v are similar. If $G[V(C_1) \cup \{u, v\}]$ has parallel edges, then $G[V(C_1) \cup \{u, v\}]$ and $G[V(C_2) \cup \{u, v\}]$ contain paths connecting u and v of equal parity. Then (G, Σ) has a $K_{2,3}^e$. Hence $G[V(C_1) \cup \{u, v\}]$ and $G[V(C_2) \cup \{u, v\}]$ are paths of length 2 and have different parity. Thus (H_1, Ω_1) is a subgraph of a sided wide 2-path where one of the parallel edges between u and v is a side.

Suppose $G_1 - \{u, v\}$ contains exactly one component. By induction (H_1, Ω_1) is a partial wide 2-path. Since in the construction of a wide 2-path parallel edges appear only parallel to the edges in \mathcal{F} of a sided wide 2-path and $(H_1, \Omega_1) - \{u, v\}$ has exactly one component, there exists a set \mathcal{F}_1 of two distinct edges, one of which is between u and v , of (H_1, Ω_1) such that $[(H_1, \Omega_1), \mathcal{F}_1]$ is a sided wide 2-path.

Similarly there exists a set \mathcal{F}_2 of two distinct edge, one of which is between the vertices of attachments of (G_2, Σ_2) such that $[(H_2, \Omega_2), \mathcal{F}_2]$ is a sided wide 2-path. Then (G, Σ) is a partial wide 2-path. We may therefore assume that G has no K_4 -minor.

Suppose G has a $K_{2,3}$ -minor. Then, by Lemma 6, (G, Σ) has a wide separation $[G_1, G_2]$, where G_1 is isomorphic to K_2^e . Let (H_2, Ω_2) be obtained from $(G_2, E(G_2) \cap \Sigma)$ by adding between the vertices attachment of $(G_2, E(G_2) \cap \Sigma)$ in the wide separation $[G_1, G_2]$ an odd and even edge in parallel. Then (H_2, Ω_2) contains no minor isomorphic to K_4^e , K_4^o , $K_{2,3}^e$, K_3^- , or W_4^o . By induction

(H_2, Ω_2) is a partial wide 2-path. Similar as above, there is a sided wide 2-path $[(J_2, \Delta_2), \mathcal{F}_2]$ such that (H_2, Ω) is a subgraph of (J_2, Δ_2) and one of the edges of \mathcal{F}_2 is an edge between the attachments of $(G_2, E(G_2) \cap \Sigma)$ in the wide separation $[G_1, G_2]$. Then (G, Σ) is a partial wide 2-path. We may therefore assume that G has no $K_{2,3}$ -minor.

Since G has no K_4 -, $K_{2,3}$ -, or K_3^- -minor, G is a partial 2-path, and so (G, Σ) is a partial wide 2-path. \square

Lemma 13. *If (G, Σ) be a 2-connected partial wide 2-path, then $M(G, \Sigma) \leq 2$.*

Proof. Suppose for a contradiction that $M(G, \Sigma) > 2$. Then there exists a matrix $A = [a_{i,j}] \in S(G, \Sigma)$ with $\text{nullity}(A) > 2$. Since (G, Σ) is a partial wide 2-path, (G, Σ) is a spanning subgraph of a wide 2-path (H, Ω) . If a wide separation in (H, Ω) is not a wide separation in (G, Σ) , then we replace the K_4^i in (H, Ω) by two adjacent triangles. We may therefore assume that each wide separation in (H, Ω) is a wide separation in (G, Σ) . For a vertex v of (G, Σ) , we denote by a_v the v th row of A .

Let s_1, s_2 be the two ends of an edge e in the set \mathcal{F} of the wide 2-path (H, Ω) . As $\text{nullity}(A) > 2$, there exists a nonzero vector $x \in \ker(A)$ with $x_{s_1} = x_{s_2} = 0$. If e belongs to a triangle, let f be the edge distinct from e in the construction of the wide 2-path (H, Ω) . Exactly one end r_1 of f belongs to $\{s_1, s_2\}$, while the other end r_2 is adjacent to one of the vertices u in $\{r_1, r_2\}$. From $a_{r_1}x = 0$ it follows that $x_{r_2} = 0$.

If e belongs to a K_4^i , then e belongs to an odd and even triangle of K_4^i . Let r_1 and r_2 be the vertices of this K_4^i distinct from s_1 and s_2 . By symmetry, we may assume that the edges s_1r_1 , s_2r_1 , and s_2r_2 are even and that the edge s_1r_2 is odd. Suppose $x_{r_1} > 0$. From $a_{s_1}x = 0$, it follows that $x_{r_2} > 0$. From $a_{s_2}x = 0$, it follows that $x_{r_1} < 0$. This contradiction shows that $x_{r_1} \leq 0$. In the same way, it is not possible that $x_{r_1} < 0$. Hence $x_{r_1} = 0$. From $a_{s_1}x = 0$, it then follows that $x_{r_2} = 0$.

Repeating the above shows that $x = 0$, which contradicts that x is a nonzero vector in $\ker(A)$. Thus $M(G, \Sigma) \leq 2$. \square

Theorem 14. *Let (G, Σ) be a 2-connected signed graph. Then the following are equivalent:*

- (i) $M(G, \Sigma) \leq 2$,
- (ii) $\xi(G, \Sigma) \leq 2$,
- (iii) (G, Σ) has no minor isomorphic to K_3^- , K_4^e , K_4^o , or $K_{2,3}^e$.
- (iv) (G, Σ) is a partial wide 2-path or isomorphic to W_4^o .

Proof. Since $\xi(G, \Sigma) \leq M(G, \Sigma)$, it is clear that (i) implies (ii).

Suppose (G, Σ) is a signed graph with $\xi(G, \Sigma) \leq 2$. Since $\xi(K_3^-) = \xi(K_4^e) = \xi(K_4^o) = \xi(K_{2,3}^e) = 3$, (G, Σ) has no minor isomorphic to K_3^- , K_4^e , K_4^o , or $K_{2,3}^e$. Hence (ii) implies (iii).

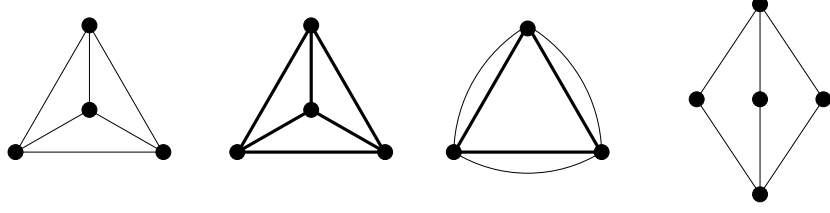


Figure 3: Forbidden minors for $\xi(G, \Sigma) \leq 2$

Suppose the signed graph (G, Σ) has no minor isomorphic to $K_3^{\bar{e}}$, K_4^e , K_4^o , or $K_{2,3}^e$. Then, by Lemma 12, (G, Σ) is either isomorphic to W_4^o or (G, Σ) is a partial wide 2-path.

If (G, Σ) is a partial wide 2-path, then, by Lemma 13, $M(G, \Sigma) \leq 2$. Since also $M(W_4^o) \leq 2$, by Lemma 4, (iv) implies (i). \square

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